Mercantile Arithmetic in Renaissance Italy: A Translation and Study of Selected Passages from a Vernacular Abbaco Work

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This essay is a study of a Renaissance Italian manuscript which has been published under the title Arte Giamata Aresmetica *('The Art Called Arithmetic'). This represents a type of mathematical text called* Libro d'Abbaco *or 'Abbacus Book', which was produced in large numbers during the Italian Renaissance (thirteenth to the sixteenth centuries CE). This paper provides the first translation into English of selected passages and an analysis of the algorithms found in this work. The style of presentation, the vocabulary used, and the mathematical procedures explained in* Arte Giamata Aresmetica *reveal some features that deserve scholarly attention.* Aresmetica *is a valuable source for the history of Renaissance business and culture as it provides insight into how merchants operated, the mathematical ability and the mind-set they had to develop, as well as revealing economic information pertaining to that period (prices, products, and various measuring units).*

THE ITALIAN TRADITION OF ABBACUS TEXTS

Previous research on the salient features of the mathematical tradition to which the text here analysed pertains is mainly written in Italian.[1] Scholarly attention to *i Libri d'abbaco* or the '*abbacu*s books' goes back to the pioneering study by the Italian historian of mathematics Boncompagni,[2] whose contributions include editions of vernacular works. Among early studies, one must mention Smith's translation of parts of a fifteenth century *abbaco* text known as *Treviso Arithmetic*, and Karpinski's short description of the *abbaco* work by Jacopo da Firenze.[3] A significant number of *abbaco* works are still extant in manuscript form in the major manuscript libraries of Italy. Recently, Arrighi, Franci, Høyrup, and Toti Rigatelli have contributed to the understanding of Renaissance Italian mathematics with important publications.[4] The first thorough study on this topic is the PhD dissertation *The Commercial Revolution and the Beginnings of Western Mathematics in Renaissance Florence, 1300–1500* by Van Egmond on which this introduction partly relies.[5] However, scholars have so far concentrated on editing *abbacus* works.[6] By contrast, this paper uses a textual approach to investigate the language and content of a specific work.

Abbacus manuscripts are a particular type of mathematical work produced in large numbers during the Italian Renaissance. Although the term *abaco/abbaco* derives from the well-known ancient counting and reckoning device, in Renaissance Italy this word came to mean 'mathematical computation' in general.[7] As pointed out by Høyrup and Van Egmond, medieval sources alternate between the spellings *abaco, abbaco,* and *abbacho*.[8] Often written in vernacular languages, their emphasis is practical and they explicated what Van Edmond calls 'business practices' for Renaissance merchants.[9] *Abbacus* texts were manuals for teachers' and merchants' training. It must be remembered that, before the introduction of the Hindu-Arabic numbers through Arabic sources, in the West the standard numeration system was based on Roman numerals. Leonardo's *Liber Abaci* (1202) is considered among the first Western books to describe the new numerical notation system based on the power of 10, which was immediately conceived of as more practical for calculation.[10] Despite some resistance in the face of this foreign invention and its use of positionality, by the beginning of the fifteenth century the Hindu-Arabic numerals had made their way through most of the Italian city-states.[11] In this regard, *abbaci* played a significant role. In some *abbaco* works, the decimal place-value system and the basic arithmetical operations are in fact the first subject treated.

The distinctive feature about this type of text is its emphasis on the practical aspect of learning: the aim of these works is to put mathematical knowledge into everyday use by practising mathematical problems of a great variety, mainly describing situations commonly occurring in the market-place of Renaissance towns. Their mathematical content varies, and includes business problems and recreational mathematics dealing with arithmetic, algebra, and geometry, as well as miscellaneous material such as calendars and astrology. As underlined by Van Egmond, the *abbaci* can be defined as collections of practical mathematical problems drawn from many areas of everyday life, but primarily from situations related to the affairs of business and commerce.[12] It is this concern for the practice of mathematics that establishes the *abbacus* tradition as a distinct field of mathematics.[13] The fact that they are written in the vernacular rather than in Latin indicates that the *abbacus* tradition belongs to the world of the market place. During the fourteenth to sixteenth centuries, *botteghe d'abbaco* or '*abbaco* schools' were flourishing, predominantly across northern Italy. The Italian word *bottega* means 'workshop', and in Renaissance society it denotes the shops where manual jobs such as weaving and dying were practised. *La bottega*, however, was also widely used for classrooms: in Italy, these institutions of learning were called *botteghe*, which is a word used to denote a shop-house of the period, usually a structure that housed a business on the ground floor and provided living quarters above.

Meastri d'abbaco probably rented such shops in which to conduct classes of instruction in them.[14] Amongst the traditions of different *abbacus* schools led by popular *abbacist*s or 'teachers of *abbaco*' which have been handed down, the most well-known are those of Giovanni di Bartolo, Filippo Calandri, and Paolo dell'Abbaco. *Abbaco* mathematics was in fact widely taught in Italian vernacular schools. Some *abbaci* explicitly state their intent to instruct children bent on commercial careers.[15] For instance, the famous Italian artist of the Renaissance, Piero della Francesca, who authored an *abbaco* text, gives the following incipit: 'Being invited to write down something on abaco, which for merchants is paramount [...].'[16]

THE MANUSCRIPT OF 'ARTE GIAMATA ARESMETICA'

Arte Giamata Aresmetica is an anonymous Renaissance Italian mathematical manuscript held in the National Library of Turin and published in 1983 by the *Quaderni del Centro Studi della Matematica Medievale*. The present essay is the first English translation and analysis of selected passages from this *abbacus* work.[17] The aim of this study is to demonstrate that, although the majority of the mathematical rules mentioned are fairly standard treatments of these subjects (not unlike those found in many other manuscripts written during this period) the uniqueness of this text lies instead in the author's expository technique, in the sample problems presented, and in its distinctive dialectical terms. *Aresmetica* was most likely a textbook for teaching commercial mathematics and merchants' skills to young pupils. I shall argue that this work is a valuable source for the history of Renaissance business and culture as it provides an insight into how merchants operated, the mathematical ability and the mind-set they had to develop, and sheds light on significant economic information of the time (prices, products, and various measuring units).[18]

In the introduction, the editor states that the manuscript is written in the Gothic script and that the author presumably hails from the northern Italian region of

Lombardy, since he often mentions the city of Milan and uses dialectical forms and various measuring units common to that area.[19] The manuscript of this text was partially destroyed by a fire in 1904. The part that survives is the beginning of the text containing eighty-seven papers. The published edition is an abridgment of the original manuscript and is presented under the title *Arte Giamata Aresmetica.* The reason for this title must be related to the incipit of the original work, where the anonymous author says that there are five names that one should learn in order to study this *arte giamata aresmetica* ('art called arithmetic').[20] This text shares some similarities with another Italian Medieval manuscript on arithmetic: *Trattato di Aritmetica e Geometria* written by Giovanni Marliani (fifteenth century CE), which is catalogued by Van Egmond under the title *Arte Giamata Arismeticha*, which is clearly a similar name to the manuscript analysed here. It is noteworthy that Marliani's text begins with a sentence all but identical to that just mentioned from the anonymous *Aresmetica*. Moreover, Marliani starts his work by listing the five names that are 'fundamental for the art called arithmetic', as *Aresmetica* also does. Rivolo emphasises that Marliani's work was probably written later than *Aresmetica* and, although it is a shorter work, it presents a larger number of mathematical topics.[21] The manuscript of *Aresmetica* can be placed around 1400 CE, on the basis of the dates found in the sample problems on interest and commercial transactions.

AN OVERVIEW OF THE MATHEMATICAL CONTENT AND THE STYLE OF 'ARESMETICA'

Aresmetica does not contain sections on addition and subtraction, and even multiplication and division are dealt with only briefly. This could denote either that simpler procedures were orally explained or that this work was meant for students already in possession of a basic knowledge of arithmetic. *Abbacists* usually followed this expository technique: the enunciation of a mathematical rule was always accompanied by a sample problem and its solution. In this manner, sample problems become practical illustrations. As in other *abbacus* manuscripts, in *Aresmetica* no mathematical symbolism is found. A few fractions and layouts in the forms of diagrams accompanying explanations randomly occur. Verbal numerical expressions are often followed by numbers in figures. No contents list or any sort of introduction, which could be a religious invocation or a kind of preface setting out the book's concepts or purpose, and which sometimes characterized other *abbaci*, are given. *Aresmetica* begins the presentation of the subject matter with no preliminaries whatsoever. The analysis of the explanatory procedures of this mathematical manual shows that this is a work with no pretensions to literary style. It conveys its information plainly and often a little awkwardly. What is clear is that this text was part of an educational tradition teaching merchant skills. In *Aresmetica*, mistakes in calculations and confusing explanations also occur, as do sample problems often given on repeated or incomplete topics. Numerical tables are also found; these were probably meant to be memorized as they are often introduced by the expression *nota bene* or 'observe', as if to say 'take note of this procedure and learn it'. The text often addresses the student by the following expression: '*fammi questa ragione*' or 'do for me this calculation'. The term *ragione* (literally 'reasoning') is widely found in *abbaci*. I do not agree with Van Egmond who translates *ragione* as 'problem', as it simply means 'reasoning, calculation'. [22]

It seems that there is a rough kind of logical progression in the topical structure of *Aresmetica*. After a brief explanation of the decimal place-value system, the text proceeds with the arithmetical operations on fractions, the rule of three, various procedures involving trade-related problems, and procedures on algebra. The topics covered in the text included:

five names related to arithmetic;

explanation of the place-value notation and of zero;

the proof by nine;

the arithmetic of fractions;

operations with mixed quantities;

the rule of three;

example of the rule of five (unnamed) (problem about monetary exchange and merchandise);

perfect numbers (and a way to know their submultiples);

rasone de compagnia (computations on various partnership problems);

rasone de barati (computations on barter and exchange of goods);

rasone de meriti (computations of non-compound interest, *meriti* denoting 'interest');

problems about alloying;

rule of false and double false position (unnamed);

rule of false position (with commercial sample problems);

computations of square-roots;

recreational mathematics;

rasone de cosa: rules and examples on algebra;

rasone de anelli: commercial computations on the value of rings and involving equations;

arithmetical games involving equations;

samples on measuring units of weight (mainly related to spices and grains);

again mixed problems on interest rates, alloying, and barter;

a kind of calendar with hours/minutes calculations;

sample problems on various topics partly incomplete and repeated.

In the present paper, I give passages in their original formulation in the footnotes, so that readers familiar with Italian vernacular can grasp and acknowledge the anachronistic style of the author, its subtle humour, and the nuances of the terminology employed. For the sake of fairness to the author, I leave the alternating use of figures and verbal expressions denoting numbers in exactly the same way as these are found in the text.

ARITHMETICAL OPERATIONS WITH MIXED QUANTITIES

The first topic expounded by *Aresmatica*'s author is the Hindu-Arabic numeral system (although this is not named as such) and operations with fractions. This feature testifies that between the fourteenth to the sixteenth centuries vernacular *abbaci* stand in close relation to the popularization of Hindu-Arabic numerals and methods of calculation.

The author of *Arte Giamata Aresmetica* begins his work listing '5 names that are fundamental in this art called arithmetic or otherwise abicho'. It can be seen that the author uses the term *abicho*, which is a vernacular variation of the word *abaco/abbaco,* while in the next page the form *abacho* occurs. The five names 'that are fundamental in this art called arithmetic or otherwise abicho' are in order: *representatione* ('representation'), *agregatione* ('aggregation' denotes the addition), *subtratione* ('subtraction'), *multiplicatione* ('multiplication'), and *divisione* otherwise called '*partitione*' ('division' or 'distribution'). What the author calls *representatione* concerns the explanation of the basic principle of the Hindu-Arabic number system: the place value notation of digits. According to Van Egmond, treatments of numeration are rare in *abbacus* manuscripts.[23] In this regard, *Aresmetica* stands out since it explains how to write and read the 10 figures. The author gives the 'representation' of the figures characterising the Hindu-Arabic numerical system.[24] The aim of the text is to highlight that by using this positional system, numbers of any size can be represented with only ten figures, instead of using the Roman long combinations of letters, from the Latin alphabet, representing numerical values. The author denotes a numerical figure as *figura*. He then explains that: 'representation has such a rule: each figure in the first place means "that many times" itself'. By 'the first place', the author means the 'units' which stand in the first place (from right to left) of the place value notation; he continues describing the tens' column: 'in the second place means [that a figure is] ten times itself', and the enumeration continues thus up to the ninth position: 'in the ninth place [each figure] means a hundred million [times] itself'. He then explains that in this system in which the digits' power proceeds from right to left in the manner of a *sarasinesca* ('shutter'), the tenth figure zero affects all the others. On this topic, *Aresmetica* gives an example and says that in the number 470 one should understand 'number, ten, and hundred' respectively. Interestingly, the author does not use the (probably later) Italian term *unità* to denote the 'unit', which is here zero, but rather the generic term *numero* or 'number'. An interesting detail is the closing phrase: *e questo basta in quanto al significare* or 'and this [which has been said] is enough in relation to the meaning'. In this context, the verb *significare* is used to denote each figure's value according to the place in the Hindu-Arabic numerical system.

The arithmetical operations with integers are not found in this work being either probably taken for granted or taught in a different context. They are instead explained in relation to computations with *i rotti* (literally 'broken'), the Italian Medieval expression denoting 'fractions'. The topic following the 'representation' of the place value system is the proof of nine in relation to multiplication. The author states that the first group of nine figures does not have any proof.[25] The proof of a ten is 1 and if one wants to know the proof of all the other remaining numbers one should *abbattere* or 'subtract' (*abbattere* is literally 'destroy') the group of nine digits and what is left is the proof of that number.[26] In this regard, let us look at the example that the text provides:

> Let it be the case that the number 23 is multiplied by 23 and we want to know its proof. I tell you that at first you should take the proof of 23, which is 5, take again [the proof of] the other 23, which is 5. Now, multiply these two proofs by each other, that is to say 5 by 5 makes 25; now take the proof of 25, which is 7 and say: 5.2.9 makes 16; subtract 9, 7 is left, so that the [result of the] multiplication and its proof will coincide. Observe and keep in mind: if you know this, you know a hundred.[27]

The sentence *sì che la multiplicatione e la prova starà bene* or 'so that the [result of the] multiplication and its proof will coincide' emphasises that the proof of the result of the multiplication and the result of the sum of the proofs of the multiplier and of the multiplicand should be the same.

After the proof of nine, the author introduces the multiplication of fractions explained only by means of sample problems and without the use of layouts showing the calculations. The forms of multiplications explained by *Aresmetica* involve *roti e sani*.[28] *Roti* and *sani* literally mean 'broken' and 'whole' respectively, and in a mathematical context denote 'fractions' and 'whole numbers'. When *sani* or 'integers' are 'broken' into parts, they become *roti* or 'fractions'. The Latin word *fractus* is the past participle of *frangere* or 'to break'. The Italian *volgare* term *roti/rotti* was at some subsequent time substituted by the Latin derived term *frazione*. Also, in this respect I would like to point out that in *abbaci* the term most commonly found is *rotti* rather than *roti*, as it occurs in Jacopo da Firenze's *Tractatus Algorism* for instance.[29] The form *roti* found in *Aresmetica* presumably represents a local dialectical variant.

In the text, the procedures on fractions listed above are not explained step-by-step, nor are layouts given or terms such as 'numerator' or 'denominator' found.[30] Instead, the author suggests multiplying the figures that are above or beneath the '*virgula*'. For instance, in relation to the first form above he says:

> Multiply 2/3 by 3/4. This is the rule: multiply the figures that are beneath the comma and say: 2 by 3 makes 6, divide it by 12, it makes 6/12, in other words 1/2. This is called to multiply fractions by fractions.[31]

It can be seen that when the product gives an improper fraction, which is a fraction with the numerator larger than the denominator, the author transforms it into a mixed quantity, hence into an integer plus a proper fraction. It is interesting to note that *Aresmetica* uses the word *virgula*, sometimes *virga,* to mean the line separating the numerator and the denominator. The same term is also found in the *abbaco* work by Jacopo da Firenze.[32] A multiplication table with fractions ends this topic.

Forms of multiplications are followed by brief explanations and sample problems on the operation of division. Four are mentioned by our *abbacus*:

1) *roti per roti* or 'a fraction by a fraction'
2) *sani per roti* or 'an integer by a fraction'
3) *roti per sani* or 'a fraction by an integer'
4) *sani e roti per sani* or 'an integer plus a fraction, by an integer'.

The text gives four other forms of division which are unnamed and involve:

5) an integer divided by an integer plus a fraction, hence using the language of the text this would be called *sani per sani e roti*;
6) an integer plus a fraction by a fraction, hence this would be *sani e roti per roti*;
7) a fraction divided by an integer plus a fraction, hence this would be *roti per sani e roti*;
8) an integer plus a fraction by an integer plus a fraction, hence this would be *sani e roti per sani e roti*.

The form enumerated above as 6 is accompanied by a layout, with the author briefly explaining that when one wants to divide it is possible, instead of inverting the position of the numerator and denominator, to simply multiply crosswise. All the exercises presented in relation to the operation of division are business problems regarding the sale of goods, such as textiles and various culinary substances.

Arithmetical practices on operations continue with the treatment of addition and subtraction of fractions, which are both explained using the crosswise method, for instance:

> This is the method for subtracting fractions, which briefly means to subtract a fraction from a fraction. Let us say that we want to subtract 1/3 from 1/2. This is the rule that you have always to keep in mind: multiply crosswise, or else 1 by 3 makes 3, then multiply 1 by 2, it makes 2. Now subtract 2 from 3, 1 remains, which is divided by [the result of] the multiplication of the figures that are beneath the *virgula*, in short 2 by 3 makes 6, thus it makes 1/6. This is also true for similar computations, where fractions are subtracted.[33]

The sample problems on addition and subtraction of fractions concern business activities and mention prices of various products. In both cases, a numerical table with fractions concludes the treatment of each operation. The verbs *sotrare* and *cavare* mean 'to subtract' and these are also found in Jacopo da Firenze's *Tractatus Algorismi*, while '*azonzere*' is 'to add'.

LA REGOLA DELLE TRE COSSE OR 'THE RULE OF THE THREE THINGS'

La regola delle tre cosse or 'the rule of the three things' (in modern mathematics known as 'the rule of three') has a special place in *abbaci* works, which usually dedicate a long section to this method.[34] It is regularly used for a great variety of problems. The rule of three involves a procedure for solving linear problems of the type, as seen in these examples:[35]

If N corresponds to Z, to what will P correspond?

The rule states that the answer is: Y= (P × Z) / N.

This is the way the author of *Aresmetica* presents this procedure:

> And now I want to explain you the rule of the three things. By this rule, you will be able to make countless mercantile calculations and many other wonderful computations. Observe the rule of the three things, since this is the crown of other rules. Let us take an example regarding the mentioned rule. For [explaining] the computation of three simple things we say: if 3 would be 6, what would 4 be? This is the rule, observe the way [by which] the divisor should be found out: that which is designated by 'if it' is the divisor, we have then to reciprocally multiply the other two parts and to divide [the result of] this multiplication by the said divisor. For instance, as I said above: if 3 would be 6, what would 4 be? Multiply 6 by 4, it makes 24, which divided by three makes 8 and this is done.Note, however, that in the rule of three it is always convenient that the first and the last thing are of the same type; that which comes out is the multiplication [whose result] is divided by the said divisor. In this way, [the result] is of the [same] type as the thing which is in the middle. Note, o sir, that your good intellect should draw attention to this helpful rule.[36] I would like to give you another instruction concerning the above said rule of the 3 things: one should ensure that among the things mentioned 3 times, the first is always the divisor and the other last 2 things should always be multiplied by each other and then divide this [result of the] multiplication by the first thing mentioned, and that which comes is [of the same type as] that which is in the middle.[37]

The author introduces this rule by emphasising its relevance for commercial practices and other calculations. The text gives straightforward and essential information: amongst the three things involved in this procedure, that which in the sample problem is introduced by the expression *se elo* or 'if it' is always the divisor. It can be understood that, though the text does not explicitly say so, the first passage refers to the rule of three when applied to problems which represent a kind of mathematical puzzle. These puzzles, which often deal with abstract numbers, can be said to be 'recreational', or alternatively 'counterfactual' calculations.[38] Conversely, in the second passage beginning with 'note, however' the text explains the rule of three when concerned with measuring units, and thus implies business situations. It appears that *Aresmetica* demands that the student pay attention to the character of each of the three terms, in order to distinguish them and carry out the procedure correctly and more quickly. The author emphasises that among 'the three things', the first and the last should be of the same type, the second is then multiplied by the last, and the result divided by the first. The final result is of the same kind as the thing in the middle, and this seems to be the meaning of the expression *si è de quela natura de quelo de mezo* and a few lines below *e quelo ch'ense è de quelo de mezo.* The exposition of this rule is followed by some counterfactual calculations involving the rule of three with fractions, hence named by the author as *la regula del 3 del roto*.

For instance, a sample problem says:

> If 2/3 were 3/4, what would 4/5 be?[39]

The author shows a layout in which two fractions are united by two diagonal lines, which indicate that one should carry out a crosswise multiplication.[40] Next, *Aresmetica* says:

> Now multiply crosswise in the same order which you see in the example and begin with the 2 that is above the 3 and say: 2 by 4 makes 8, multiply then 8 by 5, it makes 40 and this is our divisor. Multiply now the 3 that is below the 2 by the 3 that is above the 4; it makes 9. Multiply then 9 by the 4 that is above the 5, it makes 36; divide now 36 by 40, it gives 36/40, namely 18/20, or else 9/10. And it goes well. Observe [how this procedure has been solved].[41]

The text therefore explains that one should multiply crosswise the numerator 2 of the first fraction by the denominator 4 of the second fraction and by the denominator 5. This gives the divisor, the denominator (*il partitore*). Then, the denominator 3 of the first fraction is multiplied by the numerator 3 of the second fraction and by the last numerator 4. The last result is simplified reducing the fraction to lowest terms.

Thus: $(2 \times 4) \times 5 = 40$;

$(3 \times 3) \times 4 = 36$;

$\frac{36}{40} = \frac{18}{20} = \frac{9}{10}$.

Aresmetica comes up with a significant number of sample problems dealing with the rule of three, many of which draw directly from business and commercial situations. These provide interesting insights into actual commercial practices. In *abbaco* works, the most common problem-type involving the principle of proportionality is related to prices and quantities of a variety of products. Typically, the following business presentations occur: i) knowing the price of a certain quantity of a given product, one needs to find out the price of another quantity of that same product, or ii) knowing the price of a certain quantity of a given product, one needs to find out the quantity which can be bought by a given amount of money. For instance, the author gives the following sample problem:

> A *libbra* of pepper is worth 1 *ducato*. I ask you what will 7 ounces be worth? [...].[42]

The author explains the following solution:

i) convert 1 *libbra* into ounces: 1 *libbra* = 12 ounces and this is the divisor;
ii) $1 \times 7 = 7$;
iii) $7 \div 12 = 7/12$ *ducati* and this is what 7 ounces are worth.

The sample problems of this section mention weight measuring units and units from the monetary systems prevalent in different areas of Renaissance Italy. For instance:

> If 25 *braccia* of cloth are worth 29 *lira*, what will 31 *braccia* be worth? This is its rule: multiply 29 by 31, it makes 899, divide it by 25, it makes 31 *lira*, 19 *soldi*, 2 2/5 *denari* and such is what [31 *braccia*] are worth. And in this way all the similar [computations] should be done.[43]

Braccia (literally 'arm') is a unit of length widely mentioned in *abbaco* works. The abbreviation L. here stands for the monetary unit *lira*, S. stands for *soldi*, and D. for *denari*. These were account denominations with fixed ratios.[44] Though the procedure suggested by *Aresmetica* is correct, the result is not. The text says that the final result is 31 *lira,* however:

i) (29 × 31) ÷ 25 = 35, 96 lira;

ii) in *The Zibaldone da Canal,*[45] the conversion ratio amongst *lira*, *soldo*, and *denaro* is:

1 *lira* = 20 *soldi*; 1 *soldo* = 12 *denari*. By performing the calculations and acknowledging the solutions of the various sample problems, such as this very one, it becomes evident that these very same ratios also concern *Aresmetica*, since:

iii) in the quotient 35, 96 there are 35 *lira*; when the remaining 0, 96 is multiplied by 20 (1 *lira* = 20 *soldi*) the quantity of *soldi* is found, which is namely 19, 2, and when 0, 2 is multiplied by 12 (1 *soldo* = 12 *denari*) the quantity of *denari* is found, which is namely 2, 4 or else 12/5 (expressed as a mixed quantity this is 2 + 2/5).

TRADE-RELATED PROBLEMS AND MONEY EXCHANGE

It has been here already said that that business problems were the *abbacist*s' major interest. In *abbaci,* it is common to find mentioned all kinds of information that was crucial to the merchant: weights and measures of various market cities, list of currencies, products, their charge in different towns and ports, and their exchange values. Let us look, for instance, at the following *ragione*:

> A piece of cloth measuring 43 *braccia* in width and 2 [*braccia*] in height is worth 16 *ducati*. I ask you what will another piece of the same fine wool and measuring 39 *braccia* in width and 3 *braccia* in height be worth? This is its rule, which you need always to apply:

> multiply the length by the height, thus 43 by 2 makes 86, and this is the divisor. Then, multiply the [measure of the] other piece of cloth that you want to know, thus 39 by 3 makes 117, now use our rule of the three things and say: if 86 gives me 16 *ducati*, what will 117 give me? Multiply 117 by 16 and divide [the result] by 86, and the [total sum of] *ducati* [will occur] and [in this way] it goes well.[46]

This calculation involves the rule of five, which depends on the rule of three. This can be in fact expanded by adding additional terms. In *Aresmetica*, the rule of five appears to be unnamed. The author does not give the final solution of this sample problem but the computation performed is clear:

i) (43 × 2) = 86, which is the divisor;
ii) 39 × 3 = 117;
iii) (117× 16) ÷ 86 = 21, 76 *ducati*.

Abbaci provide useful information on different units of money and systems of weights and measures in the various cities of Europe. Below is a computation ('*una ragione*') on the exchange value in various Italian North cities of the length unit of measure *braccia*:

> A cloth which in Brescia is 10 *braccia* [in measure] is worth in Milan 13 *braccia* and 14 *braccia* of Mantova turn out to be in value equal to 15 *braccia* of Rimini. I want to know how much 30 *braccia* of Rimini become in Brescia.[47]

The author explains that one should:

i) multiply the first number by the third and the result by the fifth, thus:
(10×14) × 30 = 4200;
ii) multiply the second number by the fourth: 13 × 15 = 195, which is the divisor;
iii) and divide the two results: 4200 ÷ 195 = 21,538, which expressed in fractional terms is 21 7/13. This is the number of *braccia* that 3 *braccia* of Rimini become in Brescia.

The following is an interesting sample problem on products and exchange values among market cities. It is, however, also an example of how the ambiguous language used by the author and the lacking of essential details make difficult to understand his explanation and the mathematical procedure followed:

> 1234 *libbra* of wax worth in Tuscany 7 ½ *ducati* in Lombardy is sold for 7 *ducati* per C.° I ask you how much is the profit or the loss and how much would be for every C.°, warning you that one C.° of Tuscany is worth 112 [*libbre*?] in Lombardy.[48] This is its rule, we have first to find out how much [the wax] is worth in Tuscany and say: if 100 gives me 7 ½, how much will 1234 give me? That [result] which will come up is the capital. We will then say: if 100 of Tuscany becomes in Lombardy 112, how much will 1234 *libbra* of Tuscany become? That [result] which will occur is the amount of *libbra* in Lombardy. We will now reduce the mentioned *libbra* to 7 *ducati* and C.° and say: if 100 gives me 7, the *libbra* of Lombardy and that which will be found out would be the profit or the loss with respect to the capital, and that more or less would be the profit or the loss. We put this [result] aside in order to know how much we will gain or lose for [every] 100 and we say: if the capital gives me that [result] which we put aside, what will 100 give me? This [result] is what will be gained or lost for every 100. And it goes well.[49]

Below is another *ragione* giving an insight into actual commercial practices:

> A man bought clothes in France. He brought them to Milan and sold them [at the cost of] *soldi* 20 per *braccia,* earning in this way 20 per 100. I ask you how much did the piece of cloth cost?[50]

The text suggests to:

i) work out the percentage of the profit and of the capital: 20+ 100= 120;

ii) the author then observes: 'and now say: if 120 gives me 100, what will 20 *soldi* give me?'

The author suggests to apply the rule of three by keeping in mind that the first quantity is always the divisor: (100×20) ÷ 120= 16,666 or 16 *soldi* and 8 *denari*, which is the price of 1 *braccia* of cloth in France.

A further picture of a commercial situation given by *Aresmetica* is the following:

> A merchant from Pavia goes to the Saint Ambrogio fair in Milan having denari (i.e., money) in his bag. It is known that the difference between the profit and the capital is 2739 *lira* and that he earned a sum equal to 9 per 100. I ask you: what was his [initial] capital? This is its rule: say: 'if 109 gives 100, what will 2739 give?' Multiply 100 by 2739, which makes 273900, divide it by 109, it gives 2512 and 92/109. This was the amount of denari [sic] that he had as capital.[51]

Therefore in *Aresmetica* the rule of three is strained to the limits of applicability. The author tries often to accommodate the rule of three and solve the exercises by a series of proportions.

RASONE DE BARATI, RASONE DE MERITI, AND LA POXITIONE FALSA

In *Aresmetica*, *rasone de barati* or 'calculation on barter' is introduced by this remark:[52]

> This is the instruction concerning the calculation on barter (*rasone de barati*), [which is helpful] as many people are defrauded for the reason that they do not know how to make it (i.e., barter), [hence] believing to hit their [business] partners below the belt while undergoing instead this (hitting and cheating) themselves. In order to understand [this calculation], I shall propose some examples.[53]

By this statement, *Aresmetica* points out the usefulness and the practical aspect of learning the procedures explained in *abbacus* mathematics. Students were educated to develop skills necessary to all kinds of business situations. Bartering was, for instance, one of the most popular forms of commerce for Renaissance merchants.[54] The following sample problem describes the situation of two people who want to barter their products: wool and cloth. The author suggests again applying the rule of the three things:

> There are two [people] who want to barter [their products]: one has wool, the other has [a piece of] cloth. In denari (i.e., cash money) 100 [*libbra*] of wool is worth 25 *ducati*, he exchanges it for 29 [*ducati*]. The piece of cloth is in denari worth 50 *ducati*. I ask you how many *ducati* are to be exchanged in order that the barter is equal and nobody is cheated. Look, this is the rule that you have always to follow: you have to perform the rule of the three things and say in this way: if 25 gives me 29, what will 50 give me? Multiply 29 by 50, it makes 1.450, divide it by 25, it gives *ducati* 58 and such is the amount for which the piece of cloth is to be exchanged. And in this way all similar computations on barter should be done.[55]

Aresmetica provides numerous mathematical exercises dealing with barter. This topic is followed by the *rasone de meriti* or 'calculation on interest'. In the business mathematics of the *abbaci*, problems concerning finding out the amount of interest on a loan are common. Students are required to calculate the rate of interest knowing the capital and time, or to calculate the period of time necessary to earn a certain amount of interest knowing the capital and the rate of interest.[56] The author of *Aresmetica* explains the *rasone de meriti* in the following way:

> This is the instruction concerning the calculation on interest, which means that a certain amount of *lira* increments of a certain amount in a certain given time. I want to know how much that *lira* becomes in a month. This is called calculation on interest. Observe the above mentioned [calculation on] interest. Let us consider the following example: if you were told the following calculation on interest, namely that [one] *lira* is lent for a certain number of *denari* for a month and we want to know by how much 100 *lira* [increases] in one year, we have to multiply that amount of *lira* by 5, in order to know by how many *denari* a *lira* increases in a month. Let us give an example of the following rule, I shall say as follows: one *lira* is lent for 3 *denari* per month. By how much will 100 *lira* increase in a year? This is the rule, you should execute [the procedure] in this way: multiply 5 by 3 *lira*, it makes 15 *lira* and such is the increase of 100 *lira* in a year. And it goes well.[57]

This text provides more than twenty *rasone de meriti*. The following is a particular example, which is part of a group of problems defined by Van Egmond as 'equation of payments' and is often found in *abbacus* works under the name of *saldare e recare a termine* (lit. 'to pay and to finish off').[58] In fact, at the end of the given sample problem the expression used by the author of *Aresmetica* is: '*Voglio redure tuti questi presti e volio sapere a che termine debiamo pagare a tanto uno medesimo termino*...' The expression *voglio redure tuti questi presti* means 'I want to pay all these loans', while the last part of the sentence says: 'and I want to know when we need to pay [the loans all together] in a single time'. The mathematical exercise regards a merchant who lends to another merchant certain amounts of money extended over a certain period of time. The information provides a crucial evidence of the date of the manuscript:

> On February 27 1417, *lira* 50 for 1 year, 3 months, and 24 days;
>
> on April 14 1417, *lira* 60 for 1 year, 1 month, and 28 days;
>
> on June 21st 1418, *lira* 100 for 0 year, 0 month, 0 days.[59]

Aresmetica also covers another popular topic in the *abbaci*: the *regula falsi,* known as the 'rule of false position', which is based on the rule of three and is a method for solving linear equations in one or more unknowns.[60] It is commonly distinguished in the 'rule of simple position' and in the 'rule of double false position'. The *regula falsi* involves a sort of 'guess' and 'check' technique and concerns the attempt to find a solution to the problem using 'false' values to the given terms. By making a guess, the mathematician has to find out the correct answer using three variables: the guess value, the number so resulting when the guess value is applied to the conditions of the problem, and the number that we want to get. Let us look now at the way the author of *Aresmetica* explains this method, which he calls *lo modo de la poxitione falsa.* He first says that there are three ways to perform this procedure: '*a meno e da meno*', '*da più e da piú*', and '*a più e da meno*' or 'by minus and minus', 'by plus and plus', and 'by plus and minus'. These expressions describe the readjustment one needs to apply for answers either too large or too small which have been produced by the initial guesses. If the difference, i.e., the 'error' between the result wanted and the obtained is in both cases too large, one needs to subtract the answers obtained; if it is too small, one needs to sum them up. If one is too small and the other is too large one needs to subtract and add.

His exposition is both creative and functional:

> [...] false position, or else the calculation where a trick with numbers and not [a method by] reasoning occurs [...]. Keep in mind that [to perform the method of false] position means to compare [a guessed number] with the one given, or else to present a chosen number.[61] For instance, I want to divide 12 into two parts, so that the first is 4 times the second. Now let us apply the [method of] false position, or else let us make a comparison and make this question, and we will consider the first part to be 4 and the second to be 1. You can see that 4 and 1 makes 5, [but] I want 12. Now we formulate the rule of three and say: if 5 gives 12, what will 4 give? Multiply 4 by 12, it makes 48, divide it by 5, it makes 9 3/5 and so much is the first, from 9 3/5 to 12 is 2/25 and so much is the other. It is done.[62]

The author's originality continues to reveal itself: the method of false position is described as a *rasone in lo quale è ingano de numeri e non per rasone*, which emphasises that this procedure uses a kind of conjecture ('position' stands for 'supposition') and not a solution by logic.[63] By *fare de comparatione*, the author refers to the initial, arbitrary numerical guess that one must then insert into the conditions of the problems and verify if it fits. He then explains the *poxicione composita* or the rule of double false position. While modern students would probably solve this method by working out two simultaneous equations in two unknowns, according to the procedure explained by *Aresmetica* one needs to make two guesses (hence performing two 'positions') with the correct answer arising from comparison and by means of various operations working out the errors each creates.[64]The text continues with other *ragioni* such as problems on algebra and recreational problems dealing with numbers and their mathematical relationships, which for reasons of space cannot be presented here.

CONCLUSION

This paper has presented a translation and a microanalysis of selected passages from an *abbaco* vernacular work from fourteenth-century Italy. *Abbaco* was the mathematics of commerce. *Abbaci* served as textbooks for developing the mathematical skills necessary for merchants and artisans. The variety of the situations described in the sample problems, the author's style in presenting and explaining mathematical rules, the emphasis given to some rules, and the terminology used, all distinguish one *abbacus* text from another. In this regard, every *abbaco* is a unique work. These popular works explain the Hindu-Arabic numerals and methods of calculation used to solve mathematical problems from everyday life, chiefly on business and commerce.

Aresmetica presents a typical *abbacus* standard format: every rule is followed by one or more sample problems, which are always accompanied by their solution, and this contains a precise series of steps rather than merely the final result. The structure of this work follows a common pattern throughout: the author asks to perform a procedure, he then says *questa è la sua regula* or 'this is the rule', and explains how to apply the rule to the situation proposed by the sample problem. At the end of the explanation, he often adds: *nota e tene a mente* or 'observe and keep in mind'. Concluding expressions are: *e sarà fata bene* or 'and it goes well'; *è fata* or 'it is done'; and *e così vale simile* or 'and in this way all the similar [computations] should be done'. This work sets out problems dealing with weights and measures, currency exchange, interest, investments, barter, and so forth. *Aresmetica* shows that *abbaci* fully embody their surrounding culture: the culture of the merchant. The practical aspects of the *abbaci* diverge from other varieties of medieval mathematics dominated by theoretical concerns. The use of vernaculars underlines the audience for which *abbacus* texts were written: young pupils trained to be future merchants, artisans, and shopkeepers with little or no knowledge of Latin. *Aresmetica* is characterised by clumsy verbal expressions and ambiguous grammatical forms, which is a feature often occurring in other *Abbaci* too.

Notes

[1] I would like to thank the two anonymous reviewers for their helpful comments on this paper.

[2] Boncompagni published the first modern edition of Leonardo's *Liber Abaci* (1202 CE). See Baldassarre Boncompagni, *Scritti di Leonardo Pisano, matematico del secolo decimoterzo,* 2 vols (Roma: Tipografia delle Scienze Matematiche e Fisiche, 1857–1862).

[3] Smith's translation was finished and published in 1987 by Frank Swetz. See Frank J. Swetz, *Capitalism and Arithmetic: The New Math of the Fifteenth Century, including the full text of the Treviso arithmetic of 1478*, trans. by David Eugene Smith (La Salle, Ill: Open Court, 1987). Karpinski (1929) is discussed in Jens Høyrup, *Jacopo da Firenze's Tractatus Algorismi and early Italian abbacus culture* (Basel, Boston: Birkhäuser, 2007). This work is an edition, English translation, and study of a fourteenth-century *abbaco* text.

[4] Together with the Italian mathematicians Franci and Toti Rigatelli, Arrighi was the founder of the *Centro Studi della Matematica Medievale*, an academic centre in Siena for the study of medieval mathematics. It has extensively published the transcriptions into vernaculars of *abbacus* works. Arrighi dedicated himself to the publication of *abbaco* works until his death in 2001. Among his last contributions is Gino Arrighi, 'Maestro Umbro (sec.XIII), Livero de l'abbecho. Codice 2404 della Biblioteca Riccardiana di Firenze', *Bollettino della deputazione di Storia patria per l'Umbria*, 86 (1989), 5–140. Both Franci and Toti Rigatelli published summaries and transcriptions of *abbaci* under the series *Centro Studi della Matematica Medievale*.

[5] Warren Van Egmond, *The Commercial Revolution and the Beginnings of Western Mathematics in Renaissance Florence, 1300–1500* (Ann Arbor, MI: London: University Microfilms International, 1976). Van Egmond's catalogue of Italian *abbacus* manuscripts and printed books to 1600 is an important contribution towards the study of the history of this tradition. See Warren Van Egmond, *Practical Mathematics in the Italian Renaissance: A Catalogue of Italian Abbacus Manuscripts and Printed Books to 1600* (Firenze: Istituto e museo di storia della scienza, 1980).

[6] Editions of *abbaco* works are mainly in Italian. Among the few exceptions is Høyrup, *Jacopo da Firenze's Tractatus Algorismi.*

[7] The first to use the term *abaco* in this sense seems to have been Leonardo Pisano in his popular work *Liber Abaci*. Van Egmond suggests that the use of the word *Abaco* for the title of this chief work may have been the cause of the expansion and meaning attributed to this term. On Leonardo and *abbaco* texts, see Raffaella Franci, 'Leonardo Pisano e la trattatistica dell'abaco in Italia nei secoli XIV e XV', *Bollettino di Storia delle Science Matematiche*, XXIII, 2 (2003), 1–22.

[8] Høyrup, *Jacopo da Firenze's Tractatus Algorismi*, p. 3; Van Egmond, *The Commercial Revolution,* p. 5. Here I use the Anglo-Latin form *abbacus* together with its plural *abbaci* as well as the Italian form *abbaco*. Van Egmond's catalogue of Italian *abbacus* manuscripts and printed books to 1600 is a significant contribution for the understanding of the history of this mathematical tradition. See Van Egmond, *Practical mathematics in the Italian Renaissance.*

[9] Van Egmond, *The Commercial Revolution,* p. 2.

[10] On Roman numerals and their parallel use together with the newly introduced Hindu-Arabic numbers during medieval and Renaissance time, see Stephen Chrisomalis, *Numerical Notation: a Comparative History* (Cambridge: Cambridge University Press, 2010), pp. 116–127.

[11] In this regard, 'elsewhere in Western Europe, particularly in Germany and England, the Roman numerals predominated until the late fifteenth century or even later.' See Chrisomalis, *Numerical Notation,* p. 124.

[12] Van Egmond, *The Commercial Revolution,* p. 17.

[13] On the *abbaco* tradition and the development of European algebra, see Albrecht Heefer, 'The Abbaco Tradition (1300–1500): its Role in the Development of European Algebra', *Suuri Kaiseki Kenkyuujo koukyuuroku* (Japan), 30 (2008), 23–33. The algebra sections of an anonymous treatise from the *abbaco* tradition are analysed in Albrecht Heefer, 'Text production reproduction and appropriation within the abbaco tradition: a case study', *Sources and Commentaries in Exact Sciences*, 9 (2008b), 101–145.

[14] On education and learning during the Italian Renaissance, see Paul F. Grendler, *Books and Schools in the Italian Renaissance* (Aldershot: Variorum, 1995). On *abbaco* schools see Paul F. Grendler, *Schooling in Renaissance Italy: Literacy and Learning, 1300–1600* (Baltimore: Johns Hopkins University Press, 1989). Vernacular and *abbaco* schools are investigated in Robert Black, *Education and society in Florentine Tuscany* (Leiden, Boston: Brill, 2007).

[15] See Grendler, *Schooling in Renaissance Italy*, p. 311.

[16] Piero della Francesca's Trattato d'Abaco was edited by Gino Arrighi in *Trattato d'abaco, dal Codice Ashburnhamiano 280, 359–291, della Biblioteca medicea laurenziana di Firenze.* (Pisa: Domus Galilaeana, 1970), ed. by G.Arrighi.

[17] The only scholarly material available on *Aresmetica* is the brief summary and general overview in Italian in the editor's introduction. See *Arte Giamata Aresmetica: un'antologia dal codice N.III.53 della Biblioteca nazionale di Torino. Anonimo maestro lombardo*; with an introduction by Maria Teresa Rivolo (Siena: Servizio editoriale dell'Università di Siena, 1983), pp. II–XIII.

[18] The measuring units mentioned by Aresmetica are found in other texts of that period, though sometimes with different values depending on time and place. For instance, some occur in Leonardo's *Liber Abaci*, in the *Zibaldone de Canal*, in *La Pratica della Marcatura*, in Jacopo da Firenze's *Tractatus Algorismi*, in Paolo dell'Abbaco and Piero della Francesca's mathematical treatises. See *Fibonacci's Liber abaci: a translation into modern English of Leonardo Pisano's Book of calculation*, trans. by Laurence Sigler (New York: Springer, 2002); *Zibaldone da Canal: Manoscritto mercantile del sec. XIV,* ed. by Alfredo Stussi. With the contribution by F. C. Lane, Th. E. Marston, O. Ore. (Venezia: Comitato per la pubblicazione delle fonti relative alla storia di Venezia, 1967); Francesco Balducci Pegolotti, *La pratica della mercatura*, ed. by Allan Evans (Cambridge Massachusetts: Medieval Academy of America, 1936); Høyrup, *Jacopo da Firenze's Tractatus Algorismi*; *Trattato d'aritmetica by Paolo (Dagomari) dell'Abbaco, secondo la lezione del Codice Magliabechiano XI, 86 della Biblioteca nazionale di Firenze: Paolo dell' Abbaco;* ed. by G. Arrighi (Pisa: Domus Galilaeana, 1964); and Arrighi, *Trattato d'abaco*. For more details on conversion ratios and measuring units of the Italian Northern region Lombardy, see Luciana Frangioni, *Milano e le sue misure: appunti di metrologia lombarda fra Tre e Quattrocento* (Napoli: Edizioni Scientifiche Italiane, 1992).

[19] See Rivolo, *Arte Giamata Aresmetica*, p. II.

[20] In this regard, see the explanation in the next paragraph.

[21] On comparison, the vocabulary of these two works appears to share significant similarities, as well as the order of presentation of the topics. Rivolo, *Arte Giamata Aresmetica*, p. XII.

[22] Van Egmond, *The Commercial Revolution.* Høyrup, *Jacopo da Firenze's Tractatus Algorismi,* translates *ragione* as 'computation'.

[23] Van Egmond, *The Commercial Revolution*, p. 153.

[24] It is interesting to note that in Leonardo's *Abaci* and in Jacopo da Firenze's *Tractatus Algorismi* these are also listed at the very beginning of their works but in a right to left direction.

[25] The author uses the terms *novena* and *desena* to denote respectively the group of the first nine units (1–9) and a group of ten respectively.

[26] In modern mathematics, this procedure is called 'the casting out of nines'.

[27] See in Rivolo, *Arte Giamata Aresmetica*, pp. 2–3: 'Poniamo asempio che fusse multiplicato 23 via 23 e noi volesemo sapere la prova soua. Io te amagistro che prima tole la prova de ch'è 5, ancora tole la prova del altro 23 ch'è 5; ora multiplica queste due prove l'una p[er] l'a[l]tra, zoè 5 via 5 fa 25; ora piglia la prova de 25 che è 7 e dirai: 5.2.9 fa 16; abate 9, romane 7; sì che la multiplicatione e la prova starà bene. Nota e tene a mente: se tu sa questa, tue ne sa cento'.

[28] I use the term 'form' here instead of 'method' as the author does not explain different *modus operandi* for the operation of multiplication, but rather different combinations of quantities that a multiplication can involve:

1) *roti per roti* or 'a fraction [multiplied] by a fraction'
2) *sani e roti per sani e roti* or 'an integer plus a fraction by an integer plus a fraction'
3) *sani e roti per roto* 'an integer plus a fraction, by a fraction'
4) *sani e roti per sani* 'an integer plus a fraction, by an integer'
5) *sani e roti e sani* 'an integer plus a fraction plus an integer'
6) *sani e sani e roti* 'an integer plus an integer plus a fraction'

[29] In his analysis of *abbaco* manuscripts, Van Egmond says that *rotti* is the term more commonly used to denote a 'fraction'.

[30] The denominator is sometimes called *partitore* or 'divider'.

[31] See in Rivolo, *Arte Giamata Aresmetica*, p. 3: 'Multiplicare 2/3 via 3/4. Questa è la sua regula: multiplica le figure che sono de soto da le virgule e di': 2 via 3 fa 6, parte p[er] 12, vene 6/12, zoè ½. E questo s'apela multiplica roti per roti.'

[32] See Høyrup, *Jacopo da Firenze's Tractatus Algorismi*, p. 234.

[33] See in Rivolo, *Arte Giamata Aresmetica,* p. 8: '[Q]uesto è lo magistramento de sotrare de roto, zoè cavare uno roto de uno altro roto. Ponamo che volessimo cavare 1/3 de 1/2 . Questa è la sua regula, la quale tu di' sempre tenire: multiplica in crose, zoè 1 via 3 fa 3, po' multiplica 1 via 2, fa 2, hora cava 2 de 3, resta 1, lo quale de' si partito per la multiplicatione de le figure che sono de soto da la virgula, zoè 2 via 3 fa 6, che vene 1/6. E così vale simile rasone, che se cavano de boto.'

[34] *Aresmetica* presents both spellings *regola/regula*, *cose/cosse*.

[35] On the rule of three in the history of mathematics, see Jens Høyrup, 'Sanskrit-Prakrit Interaction in Elementary Mathematics as Reflected in Arabic and Italian Formulations of the Rule of Three — and Something More on the Rule Elsewhere' in *Contribution to the International Seminar on History of Mathematics*, Ramjas College, Delhi University, November 19–20, 2012 (Max-Planck-Institut für Wissenschaftsgeschichte. Preprint 435, Berlin, 2012), pp. 3–22 and Sreeramula Rajeswara Sarma, 'Rule of Three and Its Variations in India' in *From China to Paris: 2000 Years Transmission of Mathematical Ideas*, ed. by Y. Dold-Samplonius, et al. (Stuttgart: Franz Steiner Verlag, 2002), pp. 133–156.

[36] The text says *notate o misere* ('note, o sir'), interestingly using the verb in the second plural person, as a form of respect. The vocative *o misere* is a vernacular form of the honorific title *messere*.

[37] See in Rivolo, *Arte Giamata Aresmetica*, pp. 10–11: '[H]ora te voglio mostrare la regola delle 3 cosse. E per questa regola porai fa' de infenite rasone de mercadintia e de molte altre bellissime rasone. Nota obtima regula del 3 cosse ch'al è corona de le altre regule. Ponano exsempio a la dita regula: per la rasone del 3 cosse simplici diremo: se 3 fusse 6, che sarebe 4? Questa è la sua regula: per trovare lo suo partitore nota: quelo che dize se elo sempre è lo partitore; poi è cosa certa che le altre 2 parte le debiamo

multiplicare l'una per l'altra e questa tale multiplicatione la debiamo partire per lo dito partitore. Verbigracia agio dito de sopra: se 3 fusse 6, che sarebe 4? Multiplica 6 via 4, fa 24, lo quale parte per 3, che vene 8 e sarà fata. Ma nota che la cosa generalmente in la regula del 3 convene sempre che la prima cosa e la ultima cossa sia d'una medesma natura; e quelo che vene fora di poi sarà tale multiplicatione e partita per lo dito partitore si è de quela natura de quelo de mezo. Notate o misere, alzate uno pocho el vostro bono inteleto a questa gentile regula. Ancora te voio dare uno bono amaistramento in la sopra dita regula del 3 cosse: che sia certo che la cosa ch'è nommata 3 volte, la prima è sempre lo partitore e le altre 2 cose ultime debieno sempre essere multiplicate l'una per l'altra e poi partirla questa multiplicatione per la prima cosa nominata; e quelo ch'ense è de quelo de mezo.'

[38] These types of mathematical problems do not represent a business or a practical situation but instead involve purely mathematical situations. In *Jacopo da Firenze's Tractatus Algorismi,* Høyrup refers to them as 'counterfactual calculations', while in *The Commercial Revolution* Van Egmond uses the expression 'recreational calculations'.

[39] See in Rivolo, *Arte Giamata Aresmetica*, p. 11: 'Se 2/3 fusse 3/4, che sarebbe 4/5?'

[40] This sample problem from *Aresmetica* is mentioned as an example of counterfactual calculations 'in the shape of rule-of three-problems' in Høyrup, *Jacopo da Firenze's Tractatus Algorismi,* p. 64.

[41] See in Rivolo, *Arte Giamata Aresmetica*, p. 12: 'ora multiplica in crose ordinatamente come vede designato in lo exsemplo e acomenza al 2 ch'e' sopra 3 e di': 2 via 4 fa 8, poi multiplica 8 via 5, fa 40 e questo sara' lo nostro partitore; ora multiplica 8 via 5, fa 40 e questo sara' lo n'stro partitore; ora multiplica 3 che sta soto al 2 via 3 che sta sopra al 4, fa 9; po' multiplica 9 via 4 che sta sopra 5, fa 36; ora parte 36 per 40, ne vene 36/40, voia dire 18/20, zoe' 9/10. E sarà fata bene. Nota.'

[42] See in Rivolo, *Arte Giamata Aresmetica*, p. 13: '[L]a L. of pevero vale ducato 1; adomando che valerà onze 7[...].' *Libbra* was a unit of weight, while *ducato* denotes unit of money. On money and its use in medieval Europe, see Peter Spufford, *Money and its Use in Medieval Europe* (Cambridge: Cambridge University Press, 1988).

[43] See in Rivolo, *Arte Giamata Aresmetica*, p. 13: 'Se braza 25 de pano valeno L.29, che valerà braza 31? Questa è la sua regola: multiplica 29 via 31 fa 889, parte per 25, vene L.31 S.19 D. 2 2/5 e tanto vale. E così dà le simile.'

[44] The difference between a unit of account and a medium of exchange/store of wealth characterising medieval Europe is investigated in Spufford, *Money and its Use in Medieval Europe.* A study on money and banking in medieval and Renaissance Italy is by Frederic J. Lane and Reinhold C. Mueller, *Money and banking in medieval and Renaissance Italy* (Baltimore: Johns Hopkins University Press, 1985). The simultaneous use on the one hand of *lira*, *soldi*, and *denari* as monies of account and on the other hand of a *denaro* as an actual type of coin issued in silver is mentioned in Richard A. Goldthwaite, *The Economy of Renaissance Florence* (Baltimore, Md.: Johns Hopkins University Press, 2009).

[45] This is an anonymous fourteenth-century merchants' manual whose language and data suggest a Venetian author. See the English translation of this work in John E. Dotson, *Merchant Culture in Fourteenth Century Venice: the Zibaldone da Canal* (Binghamton, N.Y.: Center for Medieval and Early Renaissance Studies, State University of New York at Binghamton, 1994).

[46] See in Rivolo, *Arte Giamata Aresmetica*, p. 16: 'La peza del pano, la quale è longa 43 e alta 2 braza, vale duc.16; adomando che valerà una altra peza ch'è longa braza 39 e alta braza 3, de quela fineza medesima lana. Questa è la sua regula, tu dí sempre tenire: multiplica la longeza per l'alteza, zoè 43 via 2 fa 86 e questo è lo partitore, po' multiplica l'altra peza la quale voi sapere, zoè 39 via 3 fa 117; ora fa per la nosra regula del 3 cosse e dì: se 86 me dà duc.16, che me darà 117? Multiplica 117 via 16 e parte per 86; e quelo che vene sarano duc. e sarà bene fata.'

[47] See in Rivolo, *Arte Giamata Aresmetica*, p. 9: '[L]e braza 10 de pano de Bressa torna a Mantova braza 13 e braza 14 da Mantova torna a Rimine braza 15; volso sapere braza 30 da Rimine quanta braza da Bressa sarano. Questa è la sua regula: multiplica la prima via la terza, zoè 10 via 14 fa 140, via la quinta, 30, e 30 fa 4.200 e po' multiplica la seconda via la quarta, zoè 13 via 15 fa 195 e questo è lo partitore; ora parte 4.200 per 195, vene braza 21 7/13; e tante braza sarano da Bressa 30 da Rimine. Ed è fata bene.'

[48] I wonder whether the abbreviation C.° may stand for *cantaro*, a medieval unit of weight and volume mentioned in Leonardo's *Liber Abaci.*

[49] See in Rivolo, *Arte Giamata Aresmetica*, p. 18: 'Le L. 1.234 de cira che vale in Toscana duc.7 1/2, portandola in Lombardia donde se vende duc. 7 lo C.°, adomando che aguadagnarano overo perderano e quanto per C.°, avixandote che lo C.°de Toscana tornano in Lombardia 112. Questa è la sua regula: prima de bramo vedere quanto vale in Toscana bela e dire: se 100 me dà 7 1/2, che me darà 1.234? E quelo che verà sarà lo capitale. Po' diremo: se 100 de Toschana torna in Lombradia 112, che me tornerà L.1234 de Toscana? E quela che ne verà sarà tante L. in Lombardia. Ora reduremo le dite L. a duc.7 e C.° e diremo: se 100 me dà 7 le L.de Lombradia e quelo che verà sarà più o meno del capitale; e quelo più o meno sarà lo guadagno o la perdia; e ponamo da parte per sapere quanto per 100 aguadagnaremo ho perdaremo e diremo: s[e] el capitale me dà quelo che ponisse da parte, che me darà 100? E tanto aguadagnarono o perderano per 100. E sarà fata bene.'

[50] See in Rivolo, *Arte Giamata Aresmetica*, p. 19: '[U]no compra pagni in Franza e fali conduri a Milano; e vende lo brazo del pano S.20 e trova a gadagnare 20 per 100; adomando che li costa lo brazo del pano di capitale. Questa è la sua regula: soma 20 e 100, ch'è lo guadagno e 'l capitale: è 120, ora di': se 120 torna 100, che me tornarà S.20, ch'è lo pretio per lo quale fu venduto lo pano? Arecordete che la prima nominata è lo partitore, zoè 120: però multiplica 100 via 20, fa 2.000, parte per 120, vene S.16 D.8 e tanto costa lo dito brazo in Franza. E'fata.'

[51] See in Rivolo *Arte Giamata Aresmetica,* p. 20: '[U]no marchadante di Pavia se trova a la fera de Santo Ambroxo da Milano con D. in borsa; e trovasse infra guadagno e capitale L.2.739 e si rende certo che à guadagnato a resone de 9 per 100; adomando quale fu lo suo capitale. Questa è la sua regola: di': se 109 torna 100, che tornerà 2.739? Multiplica 100 via 2.739 che fa 273.900, parte per 109, vene 2.512 e 92/109 e tanto D. fu lo suo capitale.'

[52] Before the *rason de barati*, the author presents perfect numbers, mentioning the first two perfect numbers 6 and 28 to then gives a list of prime numbers up to 100, which he divides into two groups: 'à regula' and 'non à regula', where the last denotes prime numbers. By *hogna numero partendolo in tute quele parte che vegna numero sano* ('every number which when divided into parts comes to be a whole number') the text defines a composite number, i.e., a number which is not a prime number.

[53] See in Rivolo, *Arte Giamata Aresmetica,* p. 32: 'Questo si è lo magistramento de sapere fare rasone de barati, che de molti homini ne sono ingannati per non savere fare, credando loro di dare bota a li suoi compagni e tal hora rezeveno loro. Azochè posiamo intendere faremo exempio [...].'

[54] A thorough study on the role of merchants, money, and various forms of economic exchange is Peter Spufford, *Power and profit: the merchant in medieval Europe* (New York: Thames & Hudson, 2002).

[55] See in Rivolo, *Arte Giamata Aresmetica,* p. 33: '[...] e sono due che volevano fare barato insiema; lo uno si à lana, l'altrosi è pani; lo 100 de lana vale a D. contanti duc.25, a barato le mete 29; la peza del pano vale a D. contanti duc.50; adomando quanti duc. se de' metere a barato azochè lo barato sia e quale perchè nesuno sia inganato. Nota: questa è la sua regula, tu dì sempre tenire: farai per la regola del 3 cosse dì cossì: se 25 me dà 29, che me darà 50? Multiplica 29 via 50, fa //1.450, parte per 25, vene duc. 58 e tanto se de' metere la peza del pano a barato. E così vale le simile rasone de barato.' The abbreviation D. stands for *denari* as cash money. The merchant seems to ask a higher price if he gets goods instead of cash.

[56] See Van Egmond, *The Commercial Revolution,* pp. 178–184.

[57] See *Arte Giamata Aresmetica,* p. 33: '[Q]uesto si è lo magistramento de sapere fare raxone de meriti, como sarebe a dire: cotante L. aguagnano cotanto in cotanto tempo: adomando che vene la L. el mexe. E questo s'apela rasone de merito. Nota bene soprasorito merito. Ponamo exsemplo a la dita rasone: se al te fusse dito alcuna rasone de merito, zoè la L. è prestata el mexe acotanti D. e noi volessimo sapere che vene l'anno el 100 L., devemo moltiplicare tante L. per 5 quanti D. aguadagna la L. el mexe. Ponamo exemplo a la dita regola: dirò così: la lira è prestata el mexe D.3; che vene a guadagnare lo 100 L. l'anno? Questa è la sua regola: fa così: multiplica 5 via 3 L., fa 15 L. e tanto aguadagna 100 L. in uno anno. E sarà fata bene.'

[58] Van Egmond, *The Commercial Revolution,* p. 183.

[59] See in Rivolo, *Arte Giamata Aresmetica*, p. 34: 'a dì 27 de februaro in 1417 L.50 per anno 1, mexi 3, dì 24; a dì 14 de aprile 1417 L.60 per anno 1, mexi 1, dì 28; a dì 21 de zugno 1418 L.100 per anno 0, mexi 0, dì 0.'

[60] A study on the rule of false position in the *Liber Abaci* is by John Hannah, 'False position in Leonardo of Pisa's Liber Abbaci' in *Historia Mathematica*, 34 (2007), pp. 306–332.

[61] Here the language of the text is difficult to understand and often shows unusual grammatical forms. I imagine that by the unusual expression *fare figura a da alcuna cossa* the author wants to say that in order to apply the method of false position, one needs to work out an arbitrarily chosen value.

[62] See in Rivolo, *Arte Giamata Aresmetica,* p. 35: '[...] poxicione falsa, zoè de' rasone in lo quale è ingano de numeri e non per rasone. [...] Nota che aposicione è uno fare de comparatione a la preposita data, zoè fare figura a da alcuna cossa. Verbigratia e volio partire 12 e fare 2 parte, che la prima sia 4 tante la segonda. Ora faciamo per e faxa poxicione, zoè faciamo una similitudine e questa domanda; e diremo che la prima parte fusse 4 e la segonda fusse 1; e tu vede che 4 e 1 fano 5; io voglio 12. Ora diremo per la regula de 3 e di': se 5 me dà 12, che me darà 4? Multiplica 4 via 12, fa 48, parte per 5, che ne vene 9 3/5 e tanto a l'uno; da 9 3/5 a 12 è 2 2/5 e tanto a l'altro. E' fata.'

[63] See the double use of the word *rasone*: in the first case, *rasone* is a technical term for 'calculation', while in the second it has the literal meaning of 'reasoning'.

[64] See in Rivolo, *Arte Giamata Aresmetica,* p. 37, the interesting table provided by the text.